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On discontinuous groups acting on homogeneous spaces with non-compact isotropy subgroups

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Let G be a Lie group and H a closed subgroup. The action of a discrete subgroup Γ of G on G/H is not always properly discontinuous if H is non-compact. If the action of Γ is properly discontinuous, then Γ is called a discontinuous group acting on G/H. If G/H is of reductive type, it is known that there are *no* infinite discontinuous groups acting on G/H (called Calabi–Markus phenomenon) iff $\mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, H$. For a better understanding of discontinuous groups we are thus interested in cases (i) where G/H is non-reductive, and (ii) where G/H is of reductive type with $\mathbb{R}\text{-rank}\, G = \mathbb{R}\text{-rank}\, H + 1$. In this paper we consider the Calabi–Markus phenomenon in solvable cases of type (i). We also study discontinuous groups of reductive group manifolds for case (ii) and generalize a result of Kulkarni–Raymond to higher dimensions.

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0. Introduction

One of the basic problems in geometry has been to study how local geometric structure affects the global nature of a manifold. Our concern in this paper is with a special problem of this kind: "What is a possible fundamental group π_1 of a manifold which is locally isomorphic to a particular homogeneous space ?" This is similar to a well-studied problem in differential geometry about a possible fundamental group π_1 of a manifold under certain curvature conditions. Here are some typical examples:

(1) In the physics of relativistic cosmology, the space–time continuum is taken to be a Lorentz manifold M^4. Here a Lorentz manifold M^n is an n-dimensional manifold which bears a pseudo-Riemannian metric of type $(n -$

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1, 1). The manifold M is said to be complete if every geodesic can be defined on all time intervals. A relativistic spherical space form is a complete Lorentz manifold M^n for $n \geq 3$ with constant curvature $K = +1$. It is a remarkable result due to Calabi–Markus that every relativistic space form is non-compact and has a finite fundamental group π_1 [C-M].

(2) A Clifford–Klein form of a connected and simply connected Riemannian manifold M is a Riemannian manifold whose universal Riemannian covering is isomorphic to M. For example, any compact Riemann surface of genus ≥ 2 is regarded as a compact Clifford–Klein form of the Poincaré plane. More generally, there always exists a compact Clifford–Klein space form of a Riemannian symmetric space of the noncompact type [Bo,B-H,M-T].

(3) An affine manifold M^n is a manifold which admits a torsion free affine connection whose curvature tensor vanishes. It is called Auslander's conjecture that the fundamental group π_1 of any compact complete affine manifold is virtually solvable (see refs. [A,Mi,Ma] for instance).

These cases can be reformulated in the context of discontinuous groups acting on homogeneous spaces as follows. Let G be a Lie group and H a closed subgroup of G. A subgroup Γ of G is called *a discontinuous group acting on a homogeneous space* G/H if the action of Γ on G/H from the left is properly discontinuous. A discontinuous group acting on G/H is automatically discrete in G, whatever H is. A distinguishing feature in our setting is that H is non-compact, and consequently, a discrete subgroup is not necessarily a discontinuous group acting on G/H. This is the primary difficulty in our study. On the other hand, in the above definition of a discontinuous group we do not require freeness of the action. A small price to pay is that the double coset space $\Gamma\backslash G/H$ is not necessarily a manifold but only a V-manifold in the sense of Satake [Sa]. However, if there exists a cocompact discontinuous group Γ acting on G/H (i.e., a discontinuous group acting on G/H such that $\Gamma\backslash G/H$ is compact), then we can replace Γ by a subgroup Γ' of finite index in Γ so that $\Gamma'\backslash G/H$ is a compact smooth manifold by virtue of the result in ref. [Se]. Now the above examples are reformulated respectively as follows:

(1′) Any discontinuous group acting on $SO(n, 1)/SO(n-1, 1)$ is finite.

(2′) There exists a cocompact discontinuous group acting on G/K if G is a real linear semisimple Lie group and if K is a maximal compact subgroup of G.

(3′) Any cocompact discontinuous group acting on $GL(n,\mathbb{R}) \ltimes \mathbb{R}^n/GL(n,\mathbb{R})$ is conjectured to be virtually solvable.

Here are some comments on recent progress on (1′), (2′) and (3′).

Conjecture (3′) remains open except for some special cases such as $O(n) \ltimes \mathbb{R}^n/O(n)$ (Bieberbach's theorem, see ref. [R], corollary 8.26), $O(n, 1) \ltimes \mathbb{R}^n/O(n, 1)$ [G-K], $G \ltimes \mathbb{R}^n/G$ where G is a subgroup of $GL(n,\mathbb{R})$ which is locally isomorphic to a direct product of semisimple Lie groups of rank 1 [To].

It also remains open to classify the homogeneous spaces of reductive type (see section 3 for definition) that admit compact Clifford–Klein space forms [see (2′)]. Partial results have been obtained in refs. [Bo,M-T,Ku,Ko1,Ko3].

The feature in (1′) without infinite discontinuous groups is called *Calabi–Markus phenomenon*. In a previous paper we have established a criterion for the Calabi–Markus phenomenon in the case of a homogeneous space of reductive type:

Fact 0.1 (see refs. [C-M,Wo1,Wo2,Wo3,Ku,Ko1]). Let G/H be a homogeneous space of reductive type (see section 3 for definition). Then the following conditions are equivalent:

(i) Any discontinuous group acting on G/H is finite.

(ii) $\mathbb{R}$-rank G = $\mathbb{R}$-rank H.

In view of this, we wish to proceed a step further by posing the following questions:

Question 0.2. Suppose G/H is *not* of reductive type. Find a condition that G/H admits an infinite discontinuous group.

Question 0.3. Suppose G/H is of reductive type with $\mathbb{R}$-rank G − $\mathbb{R}$-rank $H = 1$. What can we say about a possible infinite discontinuous group acting on G/H ?

In answer to question 0.2 for solvable homogeneous spaces, we shall prove

Theorem 1 (see section 2). *Suppose G is a solvable Lie group and H is a proper closed subgroup of G. Then there exists a discrete subgroup Γ of G acting on G/H properly discontinuously and freely such that the fundamental group $\pi_1(\Gamma\backslash G/H)$ is infinite.*

This result is in sharp contrast to the reductive case; For example, the following homogeneous spaces $G/H = GL(n,\mathbb{C})/GL(n,\mathbb{R})$, $GL(m+n,\mathbb{R})/GL(m,\mathbb{R}) \times GL(n,\mathbb{R})$, $U(p,q)/SO(p,q)$, which are of the reductive type, do not admit infinite discontinuous groups by fact (0.1).

Given a subgroup Φ of G and a homomorphism $\rho : \Phi \to G$, we form a subgroup of $G \times G$ as

$$\Phi(\rho) := \{(\gamma, \rho(\gamma)) : \gamma \in \Phi\} \quad (\subset G \times G).$$

If the homomorphism ρ is the trivial representation **1**, then the action of $\Phi(\mathbf{1}) = \Phi \times 1$ on $G \simeq G \times G/\operatorname{diag} G$ is nothing but the action from the left. In this sense we might regard the action of $\Phi(\rho)$ as a "deformation" of the left action of Φ. If

Φ is a discrete subgroup of G and if the image $\rho(\Phi)$ is relatively compact, then $\Phi(\rho)$ is also a discontinuous group acting on the group manifold $G \times G/\operatorname{diag} G$.

For example, suppose that $\Phi \subset \mathrm{PSL}(2,\mathbb{R})$ is the fundamental group of a compact Riemann surface M of genus g (≥ 2), and fix generators of the first homology group $H_1(M,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Then we find the moduli space of group homomorphisms from Φ to $\mathrm{SO}(2)$ to be $\mathrm{Hom}(\Phi, \mathrm{SO}(2)) \simeq \mathbb{T}^{2g}$. That is, $\lambda \in \mathbb{T}^{2g}$ defines a homomorphism $\varphi_\lambda : \mathbb{Z}^{2g} \to \mathbb{T} \simeq \mathrm{SO}(2) \subset \mathrm{PSL}(2,\mathbb{R})$, and we get a homomorphism $\rho_\lambda : \Phi \to \mathrm{PSL}(2,\mathbb{R})$ as a composition of φ_λ and $\Phi \simeq \pi_1(M) \to \Phi/[\Phi,\Phi] \simeq H_1(M,\mathbb{Z}) \simeq \mathbb{Z}^{2g}$. Then $\Phi(\rho_\lambda) = \{(\gamma, \rho_\lambda(\gamma)) : \gamma \in \Phi\}$ forms a family of cocompact discontinuous groups acting on the group manifold of $G \times G/\operatorname{diag} G$ parametrized by $\lambda \in \mathbb{T}^{2g}$.

Even though it is hopeless to classify all discontinuous groups arising in question 0.3 because it involves all discrete subgroups of a semisimple Lie group G with $\mathbb{R}\text{-rank}\, G = 1$ (i.e. discontinuous groups acting on $G/\{e\}$), we can describe some aspects of the structure of such a discontinuous group when G/H is a group manifold $G' \times G'/\operatorname{diag} G'$, where $\mathbb{R}\text{-rank}(G' \times G') - \mathbb{R}\text{-rank}(G') = 1$ (i.e. $\mathbb{R}\text{-rank}\, G' = 1$).

Theorem 2 (see corollary 3.4). *Let G be a connected non-compact reductive linear group. Then the following conditions are equivalent.*

(1) $\mathbb{R}\text{-rank}\, G = 1$.

(2) For any torsionless discontinuous group Γ acting on $G \times G/\operatorname{diag} G$, we can find a subgroup $\Phi \subset G$ and a homomorphism $\rho : \Phi \to G$ such that $\Gamma = \{(\gamma, \rho(\gamma)) : \gamma \in \Phi\}$ up to a switch of factor.

Remark 1. Kulkarni and Raymond first proved $(1) \Rightarrow (2)$ when $G = \mathrm{SL}(2,\mathbb{R})$ in their study of three-dimensional Lorentz space forms (see theorem 5.2 and introduction in ref. [K-R]). Their proof depends on the key lemma that no discontinuous group acting on $G \times G/\operatorname{diag} G$ contains an abelian subgroup $\simeq \mathbb{Z}^2$ if $G = \mathrm{SL}(2,\mathbb{R})$. However, this is not always true even if we assume G is of $\mathbb{R}$-rank 1. For example, we can show that there exists an abelian discontinuous group $\simeq \mathbb{Z}^{n-1}$ acting on $G \times G/\operatorname{diag} G$ if $G = \mathrm{SO}(n,1)$.

Remark 2. Theorem 2 leads us to a natural question about the condition on the pair Φ and ρ such that $\Phi(\rho)$ is a discontinuous group acting on the group manifold $G \times G/\operatorname{diag} G$. In the case $G = \mathrm{SL}(2,\mathbb{R})$, it is known to be necessary that Φ is discrete (possibly after a switch of factor) [K-R]. It is not known to the author whether it is necessary that Φ is discrete (after a switch of factor) for a general $\mathbb{R}$-rank 1 group. On the other hand, it is sufficient for the discontinuity of $\Phi(\rho)$ that Φ is discrete and ρ has a relatively compact image. There are a number of examples of such homomorphisms ρ. For instance, if G is a complex semisimple Lie group and Φ is arithmetic, then we can find a non-trivial homomorphism ρ into a maximal compact group of G (e.g., ref. [Z], example 5.2.12). If $G = \mathrm{SO}_0(n,1)$ and Φ is an arithmetic cocompact discrete subgroup of G such that the first Betti number b_1 of $\Phi\backslash\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ does not vanish (Thurston's conjecture, see ref. [L]), then we have a continuous family of discontinuous groups $\Psi(\rho)$ parametrized by $\rho \in \mathrm{Hom}(\Phi, \mathbb{T}^{[n/2]}) \simeq \mathbb{T}^{b_1[n/2]}$ when $G = \mathrm{SO}_0(n,1)$, as we saw for $G = \mathrm{PSL}(2,\mathbb{R})$. Finally we also remark that in the case $G = \mathrm{SL}(2,\mathbb{R})$, some other sufficient conditions for (Φ,ρ) are also known that assure the discontinuity of $\Phi(\rho)$ on $G \times G/\operatorname{diag} G$ (see ref. [G]), but it still remains open to classify all possible (Φ,ρ) such that $\Phi(\rho)$ is a cocompact discontinuous group acting on $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})/\operatorname{diag}\mathrm{SL}(2,\mathbb{R})$.

Remark 3. It is remarkable that the example in remark 2 shows that "local rigidity" fails in higher dimensions in the case where the isotropy group is not compact. To be more precise, let Φ be a finitely generated group and G a Lie group. Let $R(\Phi, G)$ be the set of all homomorphisms of Φ into G equipped with the topology of pointwise convergence. Let H be a closed subgroup of G. We define

$$R(\Phi,G,H) := u \in R(\Phi,G) : u \text{ is injective,}$$

$$u(\Phi) \text{ is a discontinuous group acting on } G/H;$$

$$R_0(\Phi,G,H) := \{u \in R(\Phi,G,H) : u(\Phi)\backslash G/H \text{ is compact}\}.$$

A homomorphism $u \in R(\Phi,G,H)$ is called *locally rigid* if the orbit of u in $R(\Phi,G,H)$ under G is open in $R(\Phi,G,H)$. If G is semisimple with trivial center and no compact factors, then local rigidity holds for any $u \in R_0(\Phi,G,\{e\})$ (or $u \in R_0(\Phi,G,K)$ where K is a maximal compact group of G) unless $G = \mathrm{PSL}(2,\mathbb{R})$ (Weil's rigidity theorem). However, in the case of $G = \mathrm{SO}_0(n,1)$, local rigidity fails for $G \times G/\operatorname{diag} G$ because two generic elements in $\{\Phi(\rho) : \rho \in \mathrm{Hom}(\Phi, \mathbb{T}^{b_1[n/2]})\} \subset R_0(\Phi, G\times G, \operatorname{diag} G)$ (with the notation of remark 2) are not conjugate under $G \times G$.

1. Preliminary results on proper actions

First of all, let us recall the definition of a proper continuous map.

Definition 1.1 (see ref. [Bou]). Let $f : X \to Y$ be a continuous map between locally compact Hausdorff spaces. f is called *proper* iff one of the following equivalent conditions holds.

(i) f is a closed map, and $f^{-1}(y)$ is compact for any $y \in Y$.

(ii) For any topological space Z, $f : X \times Z \to Y \times Z$ is a closed map.

(iii) $f^{-1}(S)$ is compact for any compact subset S of Y.

If f is a proper map, then it follows easily that a closed subset Z of X is compact iff $f(Z)$ is contained in some compact set of Y.

Definition 1.2. Suppose that a locally compact topological (Hausdorff) group G acts continuously on a locally compact Hausdorff space X. This action is called *proper* iff the map $G \times X \ni (g, x) \mapsto (x, g \cdot x) \in X \times X$ is proper. Equivalently, $G_S := \{g \in G : g \cdot S \cap S \neq \emptyset\}$ is compact for every compact subset S in X. The action is called *properly discontinuous* iff G is discrete and acts properly on X.

The following elementary lemma deals with proper actions under an equivariant map.

Lemma 1.3. *Let G_i $(i = 1, 2)$ be locally compact groups and $L_i, H_i \subset G_i$ be closed subgroups. Suppose that $f : G_1 \to G_2$ is a (continuous) homomorphism such that $f(L_1) \subset L_2$, $f(H_1) \subset H_2$. Assume that $f(L_1)$ is closed in G_2.*

(1) Assume that $L_1 \cap \operatorname{Ker} f$ is compact. If the L_2 action on G_2/H_2 is proper, then the L_1 action on G_1/H_1 is also proper.

(2) Assume that $f(G_1)H_2 = G_2$, that $G_1 \to G_2/H_2$ is an open map, and that the quotients $L_2/f(L_1)$, $f^{-1}(H_2)/H_1$ are compact. If the L_1 action on G_1/H_1 is proper, then the L_2 action on G_2/H_2 is also proper.

Remark 1.4. If G_i are (separable) Lie groups, then the first assumption $f(G_1)H_2 = G_2$ in (2) implies the second one that the map $G_1 \to G_2/H_2$ is open.

Remark 1.5. In (2), the assumption $f(G_1)H_2 = G_2$ looks very strong. However, we cannot replace this assumption by the weaker one that $G_2/f(G_1)$ is compact. For example, let $G_1 = \mathbb{R}^n$ and W be a finite subgroup of $GL(n, \mathbb{R})$. Then we form a semi-direct product $G_2 := W \ltimes \mathbb{R}^n$. Let $f : G_1 \hookrightarrow G_2$ be a natural inclusion. Fix two abelian subspaces $L_1, H_1 \subset G_1 = \mathbb{R}^n$ such that $L_1 \cap H_1 = \{0\}$ and that $w \cdot L_1 \cap H_1 \neq \{0\}$ for some $w \in W$. Define subgroups of G_2 by $L_2 := L_1$, $H_2 := H_1$, where we regard $G_1 \subset G_2$. Then L_1 acts properly on G_1/H_1, while L_2 does not act properly on G_2/H_2. This kind of situation turns up as a reduction of the case where G_i, L_i, H_i are connected reductive groups (see ref. [Ko1], theorem 4.1).

Proof of lemma 1.3.

(1) Fix any compact subset S of G_1. We want to show that the set $\{g \in L_1 : (g \cdot S \mod H_1) \cap (S \mod H_1) \neq \emptyset \text{ in } G_1/H_1\} = L_1 \cap SH_1S^{-1}$ is compact. In view of

$$f(L_1 \cap SH_1S^{-1}) \subset L_2 \cap f(S)H_2f(S)^{-1},$$

$f(L_1 \cap SH_1S^{-1})$ is contained in a compact set if L_2 acts on G_2/H_2 properly. Then $L_1 \cap SH_1S^{-1}$ is compact, since $f_{|L_1} : L_1 \to L_2$ is a proper map because it is a composition of proper maps: $L_1 \to L_1/L_1 \cap \operatorname{Ker} f \simeq f(L_1) \hookrightarrow L_2$. That is, L_1 acts on G_1/H_1 properly.

(2) As $f(L_1)$ is a closed and cocompact subgroup of L_2, L_2 acts properly iff $f(L_1)$ ($\subset L_2$) acts properly. So we may and do assume $f(L_1) = L_2$. Take a compact set S_1 of G_1 such that $f^{-1}(H_2) = S_1 H_1$. We may assume that S_1 contains the unit of G_1. Fix any compact subset S of G_2. Let us show that $L_2 \cap S H_2 S^{-1}$ is compact. The existence of a compact subset $\tilde{S}$ of G_1 such that $f(\tilde{S})H_2 \supset S$ follows from the fact that $G_1/f^{-1}(H_2)$ is homeomorphic to G_2/H_2 (see the assumptions that $G_1 \to G_2/H_2$ is an open map and $f(G_1)H_2 = G_2$). Then we have

$$f^{-1}(L_2 \cap S H_2 S^{-1}) \subset f^{-1}(L_2) \cap \tilde{S} f^{-1}(H_2) \tilde{S}^{-1}.$$

In particular, $(f_{|L_1})^{-1}(L_2 \cap S H_2 S^{-1})$ is compact if L_1 acts properly on G_1/H_1, because

$$(f_{|L_1})^{-1}(L_2 \cap S H_2 S^{-1}) \subset L_1 \cap f^{-1}(L_2) \cap \tilde{S} f^{-1}(H_2) \tilde{S}^{-1} \subset L_1 \cap \tilde{S} S_1 H_1 S_1{}^{-1} \tilde{S}^{-1}.$$

Under our assumption $f(L_1) = L_2$, we have $L_2 \cap S H_2 S^{-1} = (f_{|L_1}) \circ (f_{|L_1})^{-1}$ $(L_2 \cap S H_2 S^{-1})$ is compact. Thus L_2 acts on G_2/H_2 properly. □

2. Homogeneous spaces of solvable groups

First we recall a nice topological property of a subgroup of a solvable Lie group due to Chevalley.

Fact 2.1 [Ch]. *Let G be a one-connected (real) solvable Lie group and H be a connected subgroup of G. Then H is closed and one-connected.*

Our main theorem in this section is

Theorem 2.2. *Let G be a solvable Lie group and H a proper closed subgroup of G. Then there exists a discrete subgroup Γ of G that acts on G/H properly discontinuously and freely such that the fundamental group $\pi_1(\Gamma \backslash G/H)$ is infinite.*

If $\sharp \pi_1(G/H) = \infty$, then we can take $\Gamma = \{e\}$ and we are done. Hereafter we suppose $\pi_1(G/H)$ is a finite group. We put $G_2 := G$, $H_2 := H$, $G_1 :=$ the universal covering group of G_2 and $H_1 :=$ the connected subgroup of G_1 with the Lie algebra $\mathfrak{h}$. We write $f : G_1 \to G_2$ for the covering map. Because $\pi_1(G/H) = \pi_1(G_2/H_2) = \pi_1(G_1/f^{-1}(H_2)) = f^{-1}(H_2)/H_1$, and because $\pi_1(G/H)$ is a finite group, we can apply lemma 1.3(2) with any subgroup $L_1 \subset G_1$ and with $L_2 := f(L_1)$. Therefore, in order to prove theorem 2.2 it suffices to prove:

Theorem 2.2′. *Let G be a one-connected (real) solvable group and H be a connected proper subgroup of G. Then there exists a discontinuous group acting on G/H which is isomorphic to $\mathbb{Z}$.*

Proof. We proceed by induction on the dimension of G. Theorem 2.2′ is clear when $\dim G = 1$, namely, when $G \simeq \mathbb{R} \supset H \simeq \{0\}$. Suppose that $\dim G \geq 2$. Then there exists a connected normal subgroup N of G with $0 < \dim N < \dim G$. We will divide into two cases according as $HN \subsetneqq G$ or $HN = G$.

(I) Assume that $HN \subsetneqq G$. The subgroup HN is connected and therefore closed by fact 2.1. So $\overline{H} := H/H \cap N = HN/N$ is a proper closed subgroup of $\overline{G} := G/N$. We write the canonical projection $\pi : G \to \overline{G} = G/N$. It follows from the inductive assumption that we can find a discrete subgroup $\overline{\Gamma}$ of $\overline{G}$ such that $\overline{\Gamma}$ is isomorphic to $\mathbb{Z}$ and acts on $\overline{G}/\overline{H}$ properly. Fix an element $\gamma \in G$ such that $\pi(\gamma)$ is a generator of $\overline{\Gamma}$. Put $\Gamma := \langle \gamma \rangle$. We have $\pi(\Gamma) = \overline{\Gamma}$, and therefore $\Gamma \simeq \mathbb{Z}$ and $\Gamma \cap N = \{e\}$. On the other hand, $\overline{\Gamma}$ is discrete and so is Γ. Applying lemma 1.3(1), we have now shown that Γ acts on G/H properly discontinuously.

(II) Assume that $HN = G$. We have $G/H \simeq N/N \cap H$ and $N \cap H \subsetneqq N$. Since $\pi_1(N/N \cap H) = \pi_1(G/H) = \{e\}$, $N \cap H$ is connected. Thus $(N, N \cap H)$ satisfies the assumption of theorem 2.2′ and $\dim N < \dim G$. Therefore we can find a discrete group $\Gamma \simeq \mathbb{Z}$ of N which acts on $N/N \cap H$ from the inductive assumption. Clearly, Γ is a subgroup of G acting properly discontinuously on G/H. □

3. $\mathbb{R}$-rank 1 semisimple group manifolds

Throughout this section, we assume that G is a connected real reductive linear Lie group. First we set up notation. Let G be a real linear reductive Lie group, with real Lie algebra $\mathfrak{g}$. Given a Cartan involution θ of G, we write a Cartan decomposition of its Lie algebra as $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. Fix a maximally abelian subspace $\mathfrak{a} \subset \mathfrak{p}$. $\mathfrak{a}$ is called a maximally split abelian subspace for G. We write $W(\mathfrak{g}, \mathfrak{a})$ for the Weyl group associated to the root system of $\Sigma(\mathfrak{g}, \mathfrak{a})$. Let $\mathbb{R}\text{-rank}\, G := \dim \mathfrak{a}$, the real rank of G. Let H be a closed subgroup of G which has finitely many connected components. If there exists a Cartan involution of G which stabilizes H, then H is called *reductive in* G and G/H is called *a homogeneous space of reductive type*. In this case, H has a Cartan decomposition $H = (H \cap K)\exp(\mathfrak{h} \cap \mathfrak{p})$, and $\mathfrak{h}$ is reductive in $\mathfrak{g}$, namely, the adjoint representation $\mathfrak{h} \to \mathfrak{gl}(\mathfrak{g})$ is completely reducible. Let $\mathfrak{a}_H$ be a maximally split abelian subspace for H. Then there exists an element g of G such that $\mathrm{Ad}(g)\mathfrak{a}_H \subset \mathfrak{a}$. Put $\mathfrak{a}(H) := \mathrm{Ad}(g)\mathfrak{a}_H$, which is uniquely defined up to conjugacy of $W(\mathfrak{g}, \mathfrak{a})$.

We shall find some structure theorem of a discontinuous group acting on a group manifold $G \times G/\operatorname{diag} G$ when $\mathbb{R}$-rank $G = 1$.

Lemma 3.1. *If $\mathbb{R}$-rank $G = 1$ and $x \in G$ is a semisimple and non-elliptic element, then $L := Z_G(x)$ is a direct product of a compact group and $\mathbb{R}$.*

Proof. There exists a Cartan involution θ of G such that $\theta L = L$. Then we have a Cartan decomposition of L:

$$\varphi : (\mathfrak{l} \cap \mathfrak{p}) \times (L \cap K) \xrightarrow{\sim} L, \qquad (X, k) \mapsto (\exp X)k.$$

Let us denote by C the center of L. According to the above decomposition, we have $C = \exp(\mathfrak{c} \cap \mathfrak{p})(C \cap K)$. It follows from the assumption that $\langle x \rangle := \{x^n : n \in \mathbb{Z}\} \simeq \mathbb{Z}$ is a discrete subgroup of G and $\langle x \rangle \subset C$. Since G is a linear group, $C \cap K$ is compact and so C is a closed abelian group with at most finitely many connected components. Therefore $\dim \mathfrak{c} \cap \mathfrak{p} \geq 1$. On the other hand, $1 = \mathbb{R}\text{-rank}\, G \geq \mathbb{R}\text{-rank}\, L = \mathbb{R}\text{-rank}[L, L] + \dim \mathfrak{c} \cap \mathfrak{p}$. Thus we have $\mathbb{R}\text{-rank}[L, L] = 0$ and therefore $\mathfrak{l} \cap \mathfrak{p} = \mathfrak{c} \cap \mathfrak{p}\ (\simeq \mathbb{R})$. Hence $\varphi : (\mathfrak{c} \cap \mathfrak{p}) \times (L \cap K) \to L$ is a Lie group isomorphism, where we regard $\mathfrak{c} \cap \mathfrak{p}$ as an additive group. □

Lemma 3.2. *If $\mathbb{R}$-rank $G = 1$ and Γ is an infinite discrete subgroup of G, then there exists a compact set S of G such that $S\Gamma S^{-1} = G$.*

Proof. It is known that any infinite discrete subgroup Γ in a linear Lie group contains an element of infinite order. Fix such an element $\gamma \in \Gamma$. In order to prove lemma 3.2 it suffices to show the existence of a compact set S such that $S\langle\gamma\rangle S^{-1} = G$. Let $\gamma = \gamma_s\gamma_u$ be its Jordan decomposition (see ref. [War], proposition 1.4.3.3), where γ_s is semisimple and γ_u is unipotent such that $\gamma_s\gamma_u = \gamma_u\gamma_s$. We divide into two cases according to whether γ_s is elliptic or not.

(I) Assume that γ_s is a non-elliptic element of G. It follows from lemma 3.1 that the only unipotent element of $Z_G(\gamma_s)$ is the identity. Since $\gamma_u \in Z_G(\gamma_s)$, we have $\gamma_u = 1$. Thus $\gamma = \gamma_s$ is contained in a maximally split Cartan subgroup J. Choose a Cartan involution θ which stabilizes J. We write the corresponding Cartan decomposition $G = \exp \mathfrak{p} K$ and we write $J = TA$, where $T := J \cap K$ and $A := J \cap \exp \mathfrak{p}$. We can write $\gamma = t \exp(Y)$ where $t \in T$, $Y \in \mathfrak{a}$. Define a compact subset of G by $S := K\{\exp sY : 0 \leq s \leq 1\}$. Then $S\langle\gamma\rangle S^{-1} \supset KAK = G$.

(II) Assume that γ_s is elliptic. Then $\gamma_u \neq 1$ since $\langle\gamma\rangle = \{\gamma_s^n\gamma_u^n : n \in \mathbb{Z}\}$ is discrete in G. By the theorem of Jacobson–Morozov, there is a Lie group homomorphism $\psi : \mathrm{SL}(2,\mathbb{R}) \to G$ such that $\psi(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}) = \gamma_u$. There is a Cartan involution θ of G such that $\theta\psi(\mathrm{SL}(2,\mathbb{R})) = \psi(\mathrm{SL}(2,\mathbb{R}))$ (see ref. [He], p. 277). In particular, $A := \psi(\{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} : a > 0\})$ is a maximally split abelian subgroup of G, which is of $\mathbb{R}$-rank 1. Define a compact subset of G by $S :=$

$K\psi(\{(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}) : 0 \le x \le 1\})\overline{\langle\gamma_s\rangle}$. Then

$$\begin{aligned} S\langle\gamma\rangle S^{-1} &\supset K\psi\left(\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} : x \in \mathbb{R}\right\}\right)K \\ &\supset K\psi\,(\mathrm{SL}(2,\mathbb{R}))\,K \supset KAK = G. \end{aligned}$$ □

Theorem 3.3. *Let G be a connected reductive linear Lie group. Then the following conditions are equivalent.*

(1) $\mathbb{R}$-rank $G \ge 2$.

(2) There exist infinite discrete subgroups Γ_i *of* G *(*$i = 1, 2$*) such that* $\Gamma := \Gamma_1 \times \Gamma_2$ *acts properly discontinuously on the group manifold* $G \times G/\operatorname{diag} G$.

Proof. We may restrict ourselves to the case where G is non-compact, namely, where $\mathbb{R}$-rank $G \ge 1$.

Suppose that $\mathbb{R}$-rank $G \ge 2$. We can find abelian subspaces $\mathfrak{a}_1$, $\mathfrak{a}_2 \subset \mathfrak{a}$ such that $\dim \mathfrak{a}_i \ge 1$ and that $W(\mathfrak{g},\mathfrak{a})\cdot\mathfrak{a}_1 \cap \mathfrak{a}_2 = \{0\}$. Put $A_i := \exp \mathfrak{a}_i$; then A_1 acts properly on G/A_2 from ref. [Ko1], theorem 4.1. Take any lattices Γ_i in abelian Lie groups A_i $(i = 1, 2)$. Then Γ_1 acts properly discontinuously on G/Γ_2, or equivalently, the discrete group $\Gamma_1 \times \Gamma_2$ acts properly discontinuously on $G \times G/\operatorname{diag} G$.

Conversely, suppose that $\mathbb{R}$-rank $G = 1$. We recall that a subgroup Γ of $G \times G$ acts properly on $G \times G/\operatorname{diag} G$ iff $C\Gamma C^{-1} \cap \operatorname{diag} G$ is compact for any compact subset C of $G \times G$. In particular, if there exists a compact set C in $G \times G$ such that $C\Gamma C^{-1} = G \times G$, then Γ acts on $G \times G/\operatorname{diag} G$ properly only if G is compact. Let Γ_i $(i = 1, 2)$ be both infinite discrete subgroups of G. It follows from lemma 3.2 that there exists a compact set S of G such that $S\Gamma_i S^{-1} = G$. In particular, $(S \times S)(\Gamma_1 \times \Gamma_2)(S^{-1} \times S^{-1}) = G \times G$. Therefore the action of $\Gamma_1 \times \Gamma_2$ on $G \times G/\operatorname{diag} G$ is not properly discontinuous because G is non-compact. □

Corollary 3.4. *Let G be a connected non-compact reductive linear group. Then the following conditions are equivalent.*

(1) $\mathbb{R}$-rank $G = 1$.

(2) Any torsionless discontinuous group Γ *in* $G \times G/\operatorname{diag} G$ *is of the following form up to a switch of factor:* $\Gamma = \{(\gamma, \rho(\gamma)) : \gamma \in \Phi\}$, *with* $\Phi \subset G$ *a subgroup and with* $\rho : \Phi \to G$ *a homomorphism.*

Proof. (2) ⇒ (1) If $\mathbb{R}$-rank $G \ge 2$, then there exist discrete subgroups $\Gamma_i \simeq \mathbb{Z}^{n_i}$ $(n_i \ge 1)$ of G such that $\Gamma_1 \times \Gamma_2$ acts properly discontinuously on $G \times G/\operatorname{diag} G$ from theorem 3.3.

(1) ⇒ (2) Suppose that Γ is a torsion free discontinuous group acting on $G \times G/\operatorname{diag} G$. Let $p_j : G \times G \to G$ $(j = 1, 2)$ be natural projections to the jth factor. Let $\Gamma_j := \operatorname{Ker} p_j \cap \Gamma$ for $j = 1, 2$. Then $\Gamma_1 \times \Gamma_2$ is regarded as a subgroup of $\Gamma \subset G \times G$, and so is also a discontinuous group acting on $G \times G/\operatorname{diag} G$. It follows

from theorem 3.3 that at least one of Γ_j must be finite if $\mathbb{R}$-rank $G = 1$. We can assume Γ_1 is a finite group after switching factor if necessary. As Γ is torsion-free, a finite subgroup Γ_1 must be trivial, namely, $p_{1|\Gamma} : \Gamma \to G$ is injective. Now Γ is of the desired form if we define $\Phi := p_1(\Gamma)$ and $\rho := p_2 \circ p_{1|\Gamma}^{-1}$. □

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